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GREEN'S FUNCTION SOLUTION FOR TRANSIENT HEAT CONDUCTION PROBLEMS

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Abstract—A derivation is given of the Green's function solution for the linear, transient heat conduction equation including the m^2T term. The solution is given in a form that explicitly and separately includes five kinds of boundary conditions. The boundary conditions are the three standard ones and an additional two involving a surface film of finite heat capacity. Some suggestions regarding a numbering system are given. The important multiplicative property of Green's functions is discussed and an example of the use of Green's functions is also given.

NOMENCLATURE

b	film thickness
c	specific heat
f	nonhomogeneous boundary condition term
F	initial temperature distribution
g	volume energy source
G	Green's function
h	heat transfer coefficient
k	thermal conductivity
m	coefficient in equation (1)
n	outward-pointing normal coordinate
r	radial coordinate
$\mathbf{r}$	general coordinates
R	region
s	number of boundary conditions
s'	surface coordinate
S	surface
t	time
T	temperature
v'	volume coordinate
x, y, z	Cartesian coordinates.

Greek symbols

α	thermal diffusivity
ρ	density
τ	dummy time variable
T	defined by equation (14)
ϕ	angular coordinate.

1. INTRODUCTION

GREEN'S functions (GFs) have been utilized in the solution of transient heat conduction equations for many decades. This is demonstrated by the discussion of GFs in the classical books of Carslaw and Jaeger [1] and Morse and Feshbach [2]. The use of GFs have not been commonly used by heat transfer engineers, however. One reason for this is the lack of an accessible and lucid derivation of the GF equation and systematic treatment for diffusion-type problems. This need has been partially met by the excellent exposition in Ozisik [3]. The present paper extends the work of Ozisik by including (a) an addition term in the basic equation, and (b) two additional boundary conditions. Furthermore, additional methods of deriving the GFs are discussed and a more complete treatment of the concept of multiplication of one-dimensional (1-D) GFs is given. Some suggestions are given regarding a numbering system for exact transient heat conduction solutions.

Other important references on GFs are the books by Greenberg [4], Roach [5] and Stakgold [6]. Aizen *et al.* [7] have provided some steady-state GFs for some basic problems. A recent translation of a book by Butkovskiy [8] provides a catalog of a number of GFs. Carslaw and Jaeger [1] also list a number of GFs.

Even though GFs are not in common use by heat transfer engineers there is ample motivation for their use in linear transient conduction for basic geometries. Some advantages of GFs are discussed next. First, they are flexible and powerful. The *same* GF for a given geometry and set of homogeneous boundary conditions is a building block for (a) a space-variable initial temperature distribution, (b) time and spatially varying boundary conditions, and (c) time- and space-variable volume energy generation.

A second advantage is that a simplified, systematic solution procedure is available. The GF can be derived just once and then tabulated as partially done in refs. [1, 8]. When using a known GF, a number of steps in the derivation can be omitted. For example, eigenfunctions and eigenvalues need not be developed. The saving of effort and reduced possibility of errors are particularly noticed for two- and three-dimensional (2- and 3-D) cases.

Third, 2- and 3-D GFs can be found for many cases by simple multiplication of a 1-D GF. This is true for the rectangular coordinate system for most of the boundary conditions considered herein, provided the problem is linear, the body is homogeneous and the geometry is 'simple'. By simple geometry is meant that any boundary must be located where only one coordinate is a constant such as $x = 0$ or $y = L$ but not along a boundary defined by, say $x+y=c$. The multiplication of a GF can result in great simplification in the derivation of 2- and 3-D GFs as well as a very compact means of cataloging these GFs. This multiplication feature of GFs was not used in Carslaw and Jaeger [1], Butkovskiy [8] and other references. Ozisik [3, p. 239] does mention this feature but does not point out any restrictions. For certain 2-D cases involving cylindrical coordinates, multiplication can be used but it cannot be used at all for spherical coordinates.

Finally, the GF can be used as building blocks in some numerical techniques such as the new unsteady surface element method [9].

The objectives of this paper are to (a) present a derivation of the GF equation including a term not previously considered and two additional boundary conditions, (b) discuss methods of deriving GFs, (c) derive a number of GFs, (d) demonstrate the conditions under which it is permissible to use products of a 1-D GF, and (e) present an example of the use of a GF.

2. DERIVATION

The partial differential equation for transient heat conduction is

$$\nabla^2 T + \frac{1}{k} g(\mathbf{r}, t) - m^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{in region } R, \quad (1)$$

where the $-m^2T$ term (not included in Ozisik [3, Chap. 6]) could represent side heat losses for a fin; m^2 is a function of $\mathbf{r}$ but not t. (The $-m^2T$ term is not needed for 3-D treatment of a fin.) The thermal conductivity, k, and thermal diffusivity, α, are both constant with position, time and temperature. Any orthogonal coordinate system, $\mathbf{r}$, can be used in equation (1) and g represents a space- and time-variable volume energy term. If there is a component of g that is linearly proportional to temperature, T, it should be included in the $-m^2T$ term which could then encompass the effects of electric heating and dilute chemical reactions; in such cases m^2 could be either positive or negative.

The boundary conditions have the general form

$$k_i \frac{\partial T}{\partial n_i} + h_i T = f_i(\mathbf{r}_i, t) - (\rho c b)_i \frac{\partial T}{\partial t}, \quad (2a)$$

where the temperature, T, and its derivatives are evaluated at the boundary surface, S_i, and $\mathbf{r}_i$ denotes the boundary. The spatial derivative, $\partial/\partial n_i$, in equation (2a) denotes differentiation along an *outward* drawn normal to the boundary surface, S_i, $i = 1, 2, \ldots, s$. The heat transfer coefficient, h_i, and $(\rho c b)_i$ can vary with position on S_i but are independent of temperature and time. (The Ozisik [3, p. 210] derivation restricts h_i to being a constant for a given surface.) This boundary condition includes the possibility of a high conductivity surface film of thickness b_i. There is a negligible temperature gradient through the film and there is no heat flux *parallel* to the surface inside the film. Five different boundary conditions can be obtained from equation (2a) by setting $k_i = 0$ or k, $h_i = 0$ or h, and also $b = 0$ or nonzero. Three of these boundary conditions are

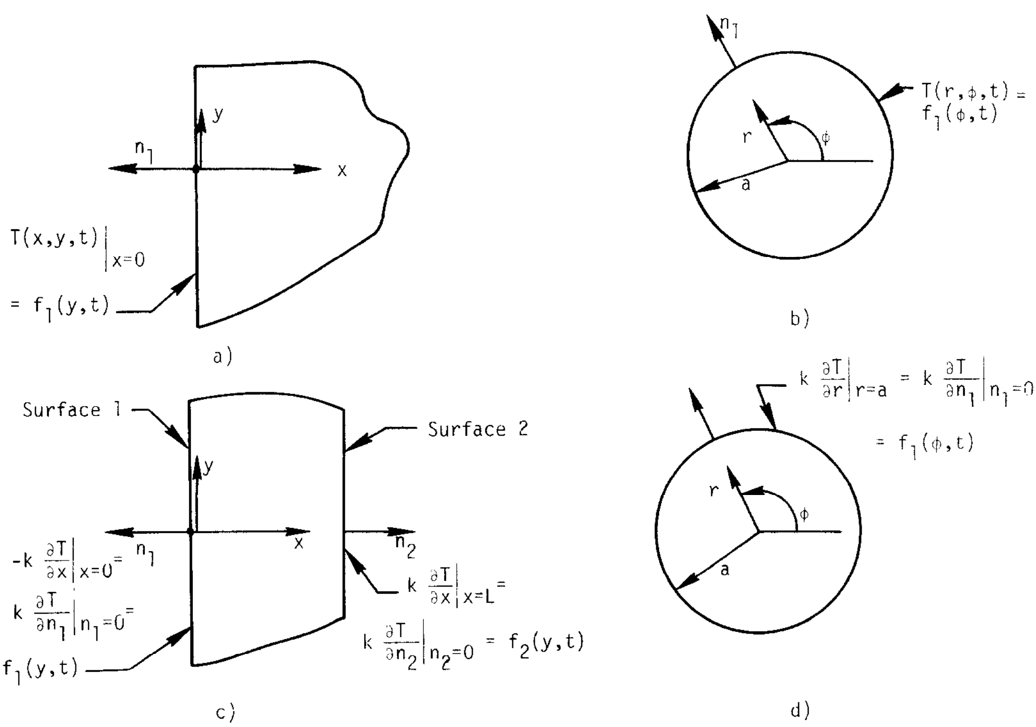

FIG. 1. Examples of boundary conditions of the first and second kinds: (a) first kind of boundary condition at $x = 0$; (b) first kind of boundary condition at $r = a$; (c) second kind of boundary condition at $x = 0$ and L, rectangular coordinates; (d) second kind of boundary condition at $r = a$, cylindrical coordinates.

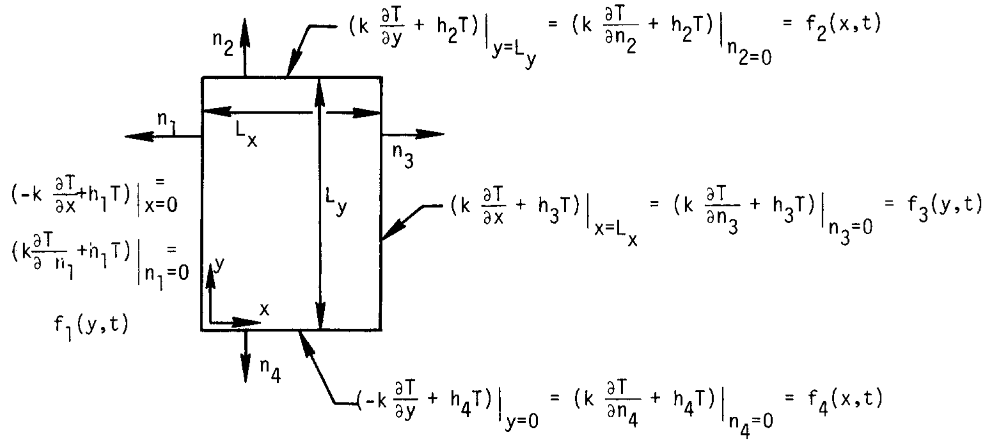

FIG. 2. Examples of convective boundary conditions (third kind).

commonly studied and are called the first, second and third kinds. The remaining two cases are called the fourth and fifth kinds.

The *first* kind of boundary condition is obtained from equation (2a) by setting $k_i = b_i = 0$ and $h_i = 1$ to get the prescribed surface temperature

$$T(\mathbf{r}_i, t) = f_i(\mathbf{r}_i, t), \tag{2b}$$

where $f_i(\mathbf{r}_i, t)$ can also be simply zero. See Figs. 1(a) and (b) for examples.

The *second* kind of boundary condition is prescribed as the surface heat flux

$$k \frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t), \tag{2c}$$

which becomes an insulation condition if $f_i(\mathbf{r}_i, t) = 0$. See Figs. 1(c) and (d).

The *third* kind is a convective boundary condition (Fig. 2)

$$k \frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} + h_i T|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t), \tag{2d}$$

and $f_i(\mathbf{r}_i, t)$ is usually $h_i T_\infty$ but could include a prescribed surface heat flux as well.

The *fourth* kind is for a thin film at the surface with no convective heat loss from the film

$$k_i \frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t) - (\rho c b)_i \frac{\partial T}{\partial t}\bigg|_{\mathbf{r}_i}. \tag{2e}$$

The *fifth* boundary includes the film and permits convective heat transfer; it is given by equation (2a). See Fig. 3.

These boundary condition numbers suggest a way to number the GF for various boundary conditions. This is discussed briefly herein and is to be developed further [10].

The initial temperature distribution is

$$T(\mathbf{r}, 0) = F(\mathbf{r}). \tag{3}$$

Consequently there are nonhomogeneous terms in the partial differential equation and the boundary conditions and there is a nonzero initial condition.

The GF equation is derived using equations (1)–(3) and also an auxiliary problem for an instantaneous heat source inside the body. This solution is given the symbol $G(\mathbf{r}, t|\mathbf{r}', \tau)$ (a GF) where the instantaneous source is located at position $\mathbf{r}'$ at time τ; $\mathbf{r}$ is the location at which the temperature is observed at time t. The *impulse* occurs at position $\mathbf{r}'$ and time τ and the *response* at $\mathbf{r}, t$. Notice that there can be a nonzero response at $\mathbf{r}$

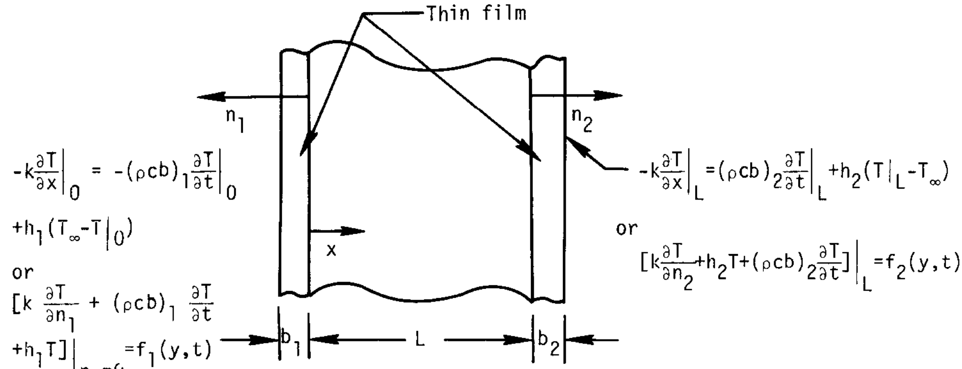

FIG. 3. Examples of film boundary condition (fifth kind).

only if $t > \tau$. The auxiliary problem has homogeneous boundary conditions and a zero initial temperature.

The derivation of the general GF equation begins using the reciprocity relation [2, p. 858]

$$G(\mathbf{r}, t|\mathbf{r}', \tau) = G(\mathbf{r}', -\tau|\mathbf{r}, -t) \quad (4)$$

(which applies for $m^2 = m^2(\mathbf{r})$) in the problem resulting in

$$\nabla_0^2 G + \frac{1}{\alpha}\delta(\mathbf{r}'-\mathbf{r})\delta(\tau-t) - m^2 G = -\frac{1}{\alpha}\frac{\partial G}{\partial \tau}, \quad (5a)$$

$$k_i \frac{\partial G}{\partial n_i} + h_i G = (\rho c b)_i \frac{\partial G}{\partial \tau}, \quad (5b)$$

$$G(\mathbf{r}', -\tau|\mathbf{r}, -t)|_{\tau=0} = 0, \quad (5c)$$

where ∇_0^2 is for the $\mathbf{r}'$ coordinates. Then equation (1) can be written in terms of $\mathbf{r}'$ and τ as

$$\nabla_0^2 T + \frac{1}{k} g(\mathbf{r}', \tau) - m^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}. \quad (6)$$

Multiply equation (6) by G and equation (5a) by T and subtract equation (5a) from equation (6) to get

$$(G\nabla_0^2 T - T\nabla_0^2 G) + \frac{1}{k} g(\mathbf{r}', \tau) G - \frac{1}{\alpha}\delta(\mathbf{r}'-\mathbf{r})\delta(\tau-t) T - m^2(GT - TG) = \frac{1}{\alpha}\frac{\partial (TG)}{\partial \tau}. \quad (7)$$

Integrate equation (7) with respect to $\mathbf{r}'$ over the region R and with respect to τ from 0 to $t^* = t+\varepsilon$, where ε is an arbitrarily small positive number. Then one obtains

$$\int_{\tau=0}^{t^*} d\tau \int_R (G\nabla_0^2 T - T\nabla_0^2 G)\, dv' + \frac{1}{k}\int_{\tau=0}^{t^*} d\tau \times \int_R g(\mathbf{r}', \tau) G\, dv' - \frac{T(\mathbf{r}, t)}{\alpha} = \frac{1}{\alpha}\int_R [GT]_{\tau=0}^{\tau=t^*}\, dv', \quad (8)$$

where dv' represents a volume element of $\mathbf{r}'$. Consider now the first term of equation (8) using Green's theorem, equation (2a) with $t \to \tau$, and equations (2b) and (5b) to get

$$\begin{aligned}
&\int_R (G\nabla_0^2 T - T\nabla_0^2 G)\, dv' \\
&= \sum_{i=1}^{s} \int_{S_i} \left(G\frac{\partial T}{\partial n_i} - T\frac{\partial G}{\partial n_i'} \right) ds_i' \\
&= \sum_{i^*=1}^{s} \int_{S_i} \left[G\left(-\frac{h_i}{k_i} T + \frac{f_i}{k_i} - \frac{(\rho c b)_i}{k_i}\frac{\partial T}{\partial \tau} \right) - T\left(-\frac{h_i}{k_i} G + \frac{(\rho c b)_i}{k_i}\frac{\partial G}{\partial \tau} \right) \right] ds_i' - \sum_{j^*=1}^{s} \int_{S_j} f_j \frac{\partial G}{\partial n_j'}\, ds_j' \\
&= \sum_{i^*=1}^{s} \int_{S_i} \left[\frac{f_i(\mathbf{r}_i', \tau)}{k_i} G(\mathbf{r}, t|\mathbf{r}', \tau)|_{\mathbf{r}'=\mathbf{r}_i} - \frac{(\rho c b)_i}{k_i} \times \frac{\partial (GT)}{\partial \tau} \right] ds_i' - \sum_{j^*=1}^{s} \int_{S_i} f_j \frac{\partial G}{\partial n_j'}\, ds_j',
\end{aligned} \quad (9)$$

which must also be integrated with respect to τ. The summation over i^* is only for the second, third, fourth and fifth kinds of boundary conditions and the j^* summation is only for j-values associated with the boundary conditions of the first kind. The reason for separating the first boundary condition from the other kinds is to avoid a division by $k_i = 0$ which would cause the first two terms to be indeterminate. The total number of terms considered between the i^* and j^* summations is exactly s, i.e. the heat flux boundary conditions (second, third, fourth and fifth kinds) and the temperature boundary condition (first kind) are mutually exclusive on a given boundary. (For a 1-D geometry, $s \leqslant 2$; for a 2-D geometry, $s \leqslant 4$; and for a 3-D geometry, $s \leqslant 6$. The number of boundary conditions, s, only includes those at 'real', finite boundaries; it does not include a boundary condition at $x \to \infty$ for a semi-infinite body, for example.)

The middle part of the last line of equation (9) yields

$$\begin{aligned}
&\sum_{i=1}^{s} \int_{S_i} \frac{(\rho c b)_i}{k_i} \int_{\tau=0}^{t^*} \frac{\partial (GT)}{\partial \tau}\, d\tau\, ds_i' \\
&= \sum_{i=1}^{s} \int_{S_i} \frac{(\rho c b)_i}{k_i} GT|_0^{t^*}\, ds_i' \\
&= -\sum_{i=1}^{s} \int_{S_i} \frac{(\rho c b)_i}{k_i} G|_{\tau=0} F(\mathbf{r}_i')\, ds_i',
\end{aligned} \quad (10)$$

since $G(\mathbf{r}, t|\mathbf{r}', t^*)$ is zero for $t^* > t$ [3, p. 213].

The RHS of equation (8) becomes

$$-\frac{1}{\alpha}\int_R G|_{\tau=0} F(\mathbf{r}')\, dv'. \quad (11)$$

Then using equations (4) and (9)–(11) in equation (8) gives

$$\begin{aligned}
&\int_{\tau=0}^{t^*} d\tau \sum_{i^*=1}^{s} \int_{S_i} \left[\frac{f_i(\mathbf{r}_i', \tau)}{k_i} G(\mathbf{r}, t|\mathbf{r}_i', \tau) \right] ds_i' \\
&- \int_{\tau=0}^{t^*} d\tau \sum_{j^*=1}^{s} \int_{S_j} f_j \frac{\partial G}{\partial n_j'}\, ds_j' \\
&+ \sum_{i^*=1}^{s} \int_{S_i} \frac{(\rho c b)_i}{k_i} G(\mathbf{r}, t|\mathbf{r}_i', 0) F(\mathbf{r}_i')\, ds_i' \\
&+ \frac{1}{k}\int_{\tau=0}^{t^*} d\tau \int_R g(\mathbf{r}', \tau) G\, dv' - \frac{T(\mathbf{r}, t)}{\alpha} \\
&= -\frac{1}{\alpha}\int_R G|_{\tau=0} F(\mathbf{r}')\, dv'.
\end{aligned} \quad (12)$$

Rearranging equation (12) then gives

$$\begin{aligned}
T(\mathbf{r}, t) &= \int_R G(\mathbf{r}, t|\mathbf{r}', 0) F(\mathbf{r}')\, dv' \\
&+ \frac{\alpha}{k}\int_{\tau=0}^{t} d\tau \int_R g(\mathbf{r}', \tau) G(\mathbf{r}, t|\mathbf{r}', \tau)\, dv' \\
&+ \alpha \int_{\tau=0}^{t} d\tau \sum_{i=1}^{s} \int_{S_i} \frac{f_i(\mathbf{r}_i', \tau)}{k} G(\mathbf{r}, t|\mathbf{r}_i', \tau)\, ds_i'
\end{aligned}$$

(boundary conditions of the second, third, fourth and fifth kinds)

$$-\alpha \int_{\tau=0}^{t} \mathrm{d}\tau \sum_{j=1}^{s} \int_{S_j} f_j(\mathbf{r}_j', \tau) \frac{\partial G}{\partial n_j'}\bigg|_{\mathbf{r}'=\mathbf{r}_j} \mathrm{d}s_j'$$

(boundary conditions of the first kind only)

$$+\alpha \sum_{i=1}^{s} \int_{S_i} \frac{(\rho cb)_i}{k} G(\mathbf{r}, t \mid \mathbf{r}_i', 0) F(\mathbf{r}_i') \,\mathrm{d}s_i', \qquad (13)$$

(boundary conditions of the fourth and fifth kinds only)

where the * superscripts are dropped for simplicity. This is the general GF equation. Space variation is permitted for m^2, $(\rho cb)_i$ and h_i.

Notice that equation (13) is for the partial differential equation given by equation (1) which includes the term $-m^2T$ which can simulate a fin or side heat losses. Surprisingly m^2 does not appear in equation (13) but the G function which is a solution of equation (5) must be a function of m^2.

The GFs are usually given for the case of $m^2 = 0$. Fortunately, these same GFs can be for the case of nonzero, spatially *constant* m^2. Let $T(\mathbf{r}, t)$ be related to $\mathrm{T}(\mathbf{r}, t)$ by

$$T(\mathbf{r}, t) = \mathrm{T}(\mathbf{r}, t) \exp[-m^2\alpha t]. \qquad (14)$$

Substitute equation (14) into equation (1) to obtain

$$\nabla^2\mathrm{T} + \frac{1}{k} g(\mathbf{r}, t) = \frac{1}{\alpha}\frac{\partial \mathrm{T}}{\partial t}, \qquad (15)$$

where the m^2 term is now missing. The eigenfunctions and values for T are exactly the same as for T. The main difference is in the treatment of the nonhomogeneous boundary term, f_i.

Consider the boundary conditions starting with the first kind

$$T(\mathbf{r}_i, t) = f_i(\mathbf{r}_i, t). \qquad (16)$$

Using the relationship given by equation (14), one obtains the T boundary condition of

$$\mathrm{T}(\mathbf{r}_i, t) = f_i(\mathbf{r}_i, t) \exp(m^2\alpha t). \qquad (17)$$

The boundary condition of the second kind

$$k \frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t), \qquad (18a)$$

can be used to get

$$k \frac{\partial \mathrm{T}}{\partial n_i}\bigg|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t) \exp(m^2\alpha t). \qquad (18b)$$

Similarly the boundary condition of the third kind

$$k \frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} = h_i(f_i(\mathbf{r}_i, t) - T(\mathbf{r}_i, t)), \qquad (19a)$$

becomes

$$k \frac{\partial \mathrm{T}}{\partial n_i}\bigg|_{\mathbf{r}_i} = h_i[f_i\, \mathrm{e}^{m^2\alpha t} - \mathrm{T}(\mathbf{r}_i, t)]. \qquad (19b)$$

The boundary conditions of the fourth and fifth kinds are more affected. Using equation (14) in equation (2a) gives

$$k_i \frac{\partial \mathrm{T}}{\partial n_i}\bigg|_{\mathbf{r}_i} + [h_i - (\rho cb)_i m^2\alpha]\mathrm{T}(\mathbf{r}_i, t) = f_i \exp(m^2\alpha t) - (\rho cb)_i \frac{\partial \mathrm{T}}{\partial t}\bigg|_{\mathbf{r}_i}. \qquad (20)$$

Notice the presence of the extra coefficient, $(\rho cb)_i m^2\alpha$, that appears in equation (20).

The initial temperature distribution given by equation (3) becomes

$$\mathrm{T}(\mathbf{r}, 0) = \mathbf{F}(\mathbf{r}), \qquad (21)$$

which is unchanged.

In summary, for the case of the transient heat conduction equation with the $-m^2T$ term with m^2 being *constant*, the GFs are exactly the same for T as for $m^2 = 0$ for boundary conditions of the first, second, third, fourth and fifth kinds. For boundary conditions of the fourth and fifth kinds h_i is replaced by $h_i - (\rho cb)_i m^2\alpha$ at the ith boundary. In addition, for each of the four boundary conditions, $f_i(\mathbf{r}_i, \tau)$ in equation (13) is replaced by $f_i(\mathbf{r}_i, \tau) \exp(m^2\alpha\tau)$. After the GF solution for $\mathrm{T}(\mathbf{r}, t)$ is obtained, $T(\mathbf{r}, t)$ is simply obtained by multiplying $\mathrm{T}(\mathbf{r}, t)$ by $\exp(-m^2\alpha t)$ as given by equation (14).

3. METHODS FOR OBTAINING GREEN'S FUNCTIONS

There are at least three different procedures for mathematically obtaining the GFs: Laplace transform method, separation of variables method, and method of images. The method in Carslaw and Jaeger [1] utilizes the Laplace transform.

The next method uses the separation of variables procedure. Ozisik [3] observed that since the same GF, $G(\cdot|\cdot)$ appears in equation (13) any part of equation (13) could be used to obtain $G(\cdot|\cdot)$. In particular, if the partial differential equation and the boundary conditions are homogeneous, equation (13) reduces to

$$T(\mathbf{r}, t) = \int_R G(\mathbf{r}, t \mid \mathbf{r}', 0) F(\mathbf{r}') \,\mathrm{d}v'. \qquad (22)$$

For the finite plate problems discussed herein, the homogeneous problem can be solved using separation of variables (see Section 5). After which, the solution is compared with equation (22) to obtain $G(\mathbf{r}, t \mid \mathbf{r}', 0)$. For the present problems $G(\mathbf{r}, t \mid \mathbf{r}', \tau)$ is simply obtained by replacing t by $t - \tau$.

The final method uses the facts that $G(\cdot|\cdot)$ is the solution for an instantaneous source and that sources and sinks can be distributed in an infinite body in such a way as to give the result for a finite body (see ref. [2, p. 863]). For an infinite body with an instantaneous plane heat source at time $t = 0$ and position x' [3, p. 220], one can write

$$G(x, t \mid x', \tau) = [4\pi\alpha(t-\tau)]^{-1/2} \times \exp[-(x-x')^2/4\alpha(t-\tau)]. \qquad (23)$$

This solution can be used to obtain $G(\cdot|\cdot)$ for a semi-infinite body with a boundary condition of the second kind at $x = 0$ (i.e. insulated) by using two sources, one at $+x'$ and the other at $-x'$

$$G(x,t|x',\tau) = [4\pi\alpha(t-\tau)]^{-1/2}\{\exp(-(x-x')^2/4\alpha(t-\tau) + \exp[-(x+x')^2/4\alpha(t-\tau)]\}. \quad (24)$$

Other cases such as $T = 0$ at $x = 0$ for a semi-infinite body and also for finite plates for boundary conditions of the first and second kinds can be obtained in a similar manner.

4. PRODUCT OF GREEN'S FUNCTIONS

One of the significant advantages of GFs for the rectangular coordinate system is that 1-D building blocks can be utilized to construct 2- and 3-D GFs by simply multiplying appropriate 1-D GFs together. This is also true for certain combinations of 2-D cases in cylindrical coordinates.

In Ozisik [3, p. 239] it is stated that multiplication of 1-D GFs is possible but restrictions are given neither regarding the specific equation covered in this paper, equation (1), nor regarding the five kinds of boundary conditions. This is also true for Carslaw and Jaeger [1, pp. 361; 33–35].

The case of Cartesian coordinates is investigated first. The objective is to prove that the GF produces the relation

$$G(x,y,z,t|x',y',z',\tau) = G_1(x,t|x',\tau)G_2(y,t|y',\tau)G_3(z,t|z',\tau). \quad (25)$$

The 3-D equation considered is

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}+\frac{\partial^2 G}{\partial z^2}-(m_1^2+m_2^2+m_3^2)G-\frac{1}{\alpha}\frac{\partial G}{\partial t} = -\frac{1}{\alpha}\delta(x-x')\delta(y-y')\delta(z-z')\delta(t-\tau), \quad (26)$$

where m_1^2 is a function of x only, m_2^2 of y only and m_3^2 of z only. To see if equation (25) satisfies equation (26) introduce equation (25) into equation (26) to get

$$G_2G_3\left[\frac{\partial^2 G_1}{\partial x^2}-m_1^2G_1-\frac{1}{\alpha}\frac{\partial G_1}{\partial t}\right] + G_1G_3\left[\frac{\partial^2 G_2}{\partial y^2}-m^2G_2-\frac{1}{\alpha}\frac{\partial G_2}{\partial t}\right] + G_1G_2\left[\frac{\partial^2 G_3}{\partial z^2}-m_3^2G_3-\frac{1}{\alpha}\frac{\partial G_3}{\partial t}\right] = -\frac{1}{\alpha}\delta(x-x')\delta(y-y')\delta(z-z')\delta(t-\tau). \quad (27)$$

The 1-D describing equations for G_1, G_2 and G_3 are

$$\frac{\partial^2 G_1}{\partial x^2}-m_1^2G_1-\frac{1}{\alpha}\frac{\partial G_1}{\partial t} = -\frac{1}{\alpha}\delta(x-x')\delta(t-\tau), \quad (28a)$$

$$\frac{\partial^2 G_2}{\partial y^2}-m_2^2G_2-\frac{1}{\alpha}\frac{\partial G_2}{\partial t} = -\frac{1}{\alpha}\delta(y-y')\delta(t-\tau), \quad (28b)$$

$$\frac{\partial^2 g_3}{\partial z^2}-m_3^2G_3-\frac{1}{\alpha}\frac{\partial G_3}{\partial t} = -\frac{1}{\alpha}\delta(z-z')\delta(t-\tau). \quad (28c)$$

If equation (28a) is multiplied by G_2G_3, equation (28b) by G_1G_3 and equation (28c) by G_1G_2 and these three equations are added one obtains the same LHS as equation (27) and the RHS of

$$-\frac{1}{\alpha}G_2G_3\delta(x-x')\delta(t-\tau)-\frac{1}{\alpha}G_1G_3\delta(y-y')\delta(t-\tau) - \frac{1}{\alpha}G_2G_2\delta(z-z')\delta(t-\tau). \quad (29)$$

It can be shown that (see Appendix B)

$$G_2G_3\delta(x-x')\delta(t-\tau) = \frac{1}{3}\delta(x-x')\delta(y-y')\delta(z-z')\delta(t-\tau), \quad (30)$$

and similar relations can be written for other terms in equation (29). Thus the 1-D equations, equations (28a)–(28c), are consistent with the 3-D equation, equation (27). Hence equation (25) satisfies the 3-D partial differential equation, equation (26).

It now remains to consider the boundary conditions. The x-direction conditions are

$$k_i\frac{\partial G}{\partial x} \mp h_iG = 0 \quad \text{at } x = 0 \text{ and } L, \quad (31)$$

and similar conditions are given for the y- and z-directions. Note that the fourth and fifth kinds of boundary conditions are *not* permitted. Also the heat transfer coefficient h_i for a given surface must be constant. The boundary condition for G_1 is written in the same form as equation (31)

$$k_i\frac{\partial G_1}{\partial x} \mp h_iG_1 = 0 \quad \text{at } x = 0 \text{ and } L. \quad (32)$$

By replacing G in equation (31) by the product $G_1G_2G_3$ and differentiating, one obtains the same expression as equation (32) multiplied by G_2G_3. Similar equations are obtained for the y- and z-directions. Then the product relation, equation (25), is valid at the boundaries. Hence, it has been shown that G is given by the product $G_1G_2G_3$ for the case of (a) Cartesian coordinates for equation (1) with the m^2 term, (b) m^2 in equation (1) can be the sum of three functions, one of x only, one of y only and the last of z only, (c) boundary conditions of the first, second and third kinds, and (d) h_i is a constant for a given boundary at $x = 0$ or L, $y = 0$, etc. The product relation is also valid if one or more boundaries are located at $x = \pm\infty$, $y = \pm\infty$ or $z = \pm\infty$.

A similar analysis can be performed for cylindrical coordinates, (r, ϕ, x), starting with the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 G}{\partial \phi^2}+\frac{\partial^2 G}{\partial x^2}-(m_r^2+m_\phi^2+m_x^2)G$$
$$-\frac{1}{\alpha}\frac{\partial G}{\partial t} = -4\pi\frac{1}{\alpha}\frac{1}{r}\delta(r-r')\delta(\phi-\phi')\delta(x-x')\delta(t-\tau), \quad (33)$$

where m_r^2 is a function of r, etc. Assume that

$$G(r,\phi,x,t) = G_1(r,t)G_2(\phi,t)G_3(x,t), \quad (34)$$

so that equation (33) can be written as

$$G_2G_3\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G_1}{\partial r}\right)-m_r^2G_1-\frac{1}{\alpha}\frac{\partial G_1}{\partial t}\right]$$
$$+G_1G_3\left[\frac{1}{r^2}\frac{\partial^2 G_2}{\partial \phi^2}-m_\phi^2G_2-\frac{1}{\alpha}\frac{\partial G_2}{\partial t}\right]$$
$$+G_1G_2\left[\frac{\partial^2 G_3}{\partial x^2}-m_x^2G_3-\frac{1}{\alpha}\frac{\partial G_3}{\partial t}\right]=-\frac{1}{\alpha}\frac{1}{r}$$
$$\times 4\pi\delta(r-r')\delta(\phi-\phi')\delta(x-x')\delta(t-\tau).$$

Since r^2 appears inside the second brackets, equation (34) is not valid. One can, however, use the product solution for the GF for r and x. Furthermore, the ϕ and x multiplication can be used if a thin shell is being considered (i.e. r = const.).

For the spherical coordinate system no multiplicative relation is possible for the 1-D GF to get 2- or 3-D GF.

5. DERIVATION OF GREEN'S FUNCTIONS FOR PLATE

For Cartesian coordinates and constant m^2 and h_i, basic cases for boundary conditions of the first, second and third kinds can be obtained from the 1-D problem involving the equation, equation (A1), in Appendix A. Products of GFs can be used for 2- and 3-D cases as mentioned above. To be more general, boundary conditions of the first–fifth kinds are considered and are given by equation (A2). A derivation is given in Appendix A. One of the unique aspects of the derivation is that a special weighting function in the orthogonality relation, equation (A9), is needed. See equation (A10).

The derivation in Appendix A is for the temperature for homogeneous boundary conditions and a homogeneous differential equation. The temperature $T(x,t)$ given by equation (A12) is related to the GF through the first term of equation (13). The GF is the summation in equation (A12) (except for $F(x')\,dx'$) with t replaced by $t-\tau$

$$G(x,t|x',\tau)=\frac{X_0(x)}{N_0}+\sum_{m=1}^{\infty}e^{-\beta_m^2(\alpha(t-\tau)/L^2)}\frac{X_m(x)X_m(x')}{N_m}. \quad (35)$$

The norms, N_0 and N_m, the eigenfunction, $X_m(x)$, and the eigencondition for obtaining β_m are given in Tables 1 and 2.

Also included in Tables 1 and 2 is a numbering system for the 1-D GFs. The GF number index begins with X for the x-coordinate and is followed by two digits, the first one for the $x = 0$ boundary and the second one for the $x = L$ boundary. That is, the numbers are denoted XIJ where, for Appendix A, I and J go from 1 to 5 and correspond to boundary conditions for the first–fifth kinds. For example, the Fig. 1(c) case has the number $X22$ and the Fig. 3 problem has the number $X55$. If there is no boundary, the digit 0 is used; for Fig. 1(a) the digit is $X10$. This number system can be extended to other coordinates and multi-dimensional cases [10].

Table 1. Proposed numbering system for GFs for plate of thickness L

Boundary conditions:

$$x=0:\quad -K_1L\frac{\partial T}{\partial x}+B_1T+C_1L^2\frac{\partial^2 T}{\partial x^2}=0;$$

$$x=L:\quad K_2L\frac{\partial T}{\partial x}+B_2T+C_2L^2\frac{\partial^2 T}{\partial x^2}=0;$$

where $K_i = k_i/k$, $B_i = h_iL/k$ and $C_i = (\rho cb)_i/\rho cL$, i = 1 and 2.
(Note that $\partial^2T/\partial x^2 = (1/\alpha)\partial T/\partial t$.)
Use XIJ, I, J = 1, 2, 3, 4, 5.

I	K_1	B_1	C_1	J	K_2	B_2	C_2
1	0	1	0	1	0	1	0
2	1	0	0	2	1	0	0
3	1	B_1	0	3	1	B_2	0
4	1	0	C_1	4	1	0	C_2
5	1	B_1	C_1	5	1	B_2	C_2

	Eigenfunctions	A_1	A_2
$X1J, J$ = 1, 2, 3, 4, 5	$\sin\beta_m x/L$	1	0
$X2J, J$ = 1, 2, 3, 4, 5	$\cos\beta_m x/L$	0	1
$X31$	$\sin\beta_m(L-x)/L$	1	0
$X32$	$\cos\beta_m(L-x)/L$	0	1
$X33, X34, X35$	$B_1\sin\beta_m x/L+\beta_m\cos\beta_m x/L$	B_1	β_m
$X4J, J$ = 1, 2, 3, 4, 5	$-C_1\beta_m\sin\beta_m x/L+\cos\beta_m x/L$	$-C_1\beta_m$	1
$X5J, J$ = 1, 2, 3, 4, 5	$(B_1-C_1\beta_m^2)\sin\beta_m x/L+\beta_m\cos\beta_m x/L$	$B_1-C_1\beta_m^2$	β_m

Table 2. Eigenvalues and norms for GFs for plate

Eigenvalues are positive roots of:

$$\tan \beta_m = \frac{\beta_m[K_1(B_2 - C_2\beta_m^2) + K_2(B_1 - C_1\beta_m^2)]}{K_1K_2\beta_m^2 - (B_1 - C_1\beta_m^2)(B_2 - C_2\beta_m^2)}.$$

Simple cases:

for $X11$ and $X22$, $\beta_m = m\pi$, $m = 1, 2, \ldots$;

for $X12$ and $X21$, $\beta_m = (2m-1)\pi/2$, $m = 1, 2, \ldots$

Norms for $m = 1, 2, \ldots$

$$N_m = L\left\{\frac{1}{2}(A_1^2 + A_2^2) + A_2^2(C_1 + C_2) + \frac{\tan \beta_m}{1 + \tan^2 \beta_m}\left[\frac{1}{2\beta_m}(A_2^2 - A_1^2) + 2C_2A_1A_2 + \tan(\beta_m)\left(C_2(A_1^2 - A_2^2) + \frac{1}{\beta_m}A_1A_2\right)\right]\right\}.$$

Simple cases: $N_m = \frac{L}{2}$ for $X11$, $X12$, $X21$, and $X22$.

Special cases: for $X22$, $X24$, $X42$, and $X44$ for $\beta_0 = 0$

$X_0(x) = 1$;

$N_0 = (1 + C_1 + C_2)L$;

for all other cases, $X_0(x) = 0$.

---

6. EXAMPLE

There are a number of examples of using GFs for boundary conditions of the first kind in Ozisik [3, Chap. 6].

A problem involving a GF for boundary conditions of the first and second kinds is shown in Fig. 4 and described mathematically by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \tag{36}$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0 = \text{const.}, \tag{37a}$$

$$T = 0 \quad \text{at} \quad x = a, \tag{37b}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0, \quad T = T_0 \quad \text{at} \quad y = b, \tag{37c,d}$$

$$T(x, y, 0) = 0. \tag{37e}$$

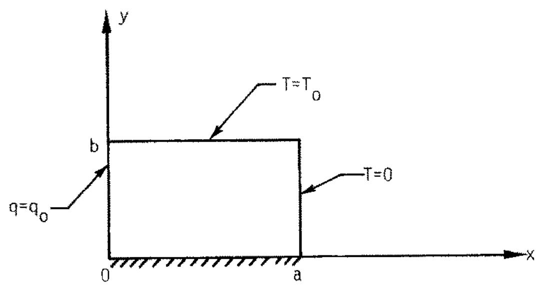

FIG. 4. Cross section of a rectangular 2-D heat conduction problem.

The 2-D GF is formed by multiplying two 1-D GFs together. The 1-D GFs are similar with a boundary condition of the second kind at $x = 0$ (or $y = 0$) and of the first kind at $z = a$ (or $y = b$), i.e. $X21$.

$$G(x, y, t|x', y', \tau) = G_X(x, t|x', \tau)G_Y(y, t|y', \tau), \tag{38a}$$

$$G_X(x, t|x', \tau) = \frac{2}{a}\sum_{m=1}^{\infty} e^{-\beta_m^2(\alpha(t-\tau)/a^2)} \times \cos\left(\beta_m\frac{x}{a}\right)\cos\left(\beta_m\frac{x'}{a}\right), \tag{38b}$$

$$G_Y(y, t|y', \tau) = \frac{2}{b}\sum_{n=1}^{\infty} e^{-\beta_n^2(\alpha(t-\tau)/b^2)} \times \cos\left(\beta_n\frac{y}{b}\right)\cos\left(\beta_n\frac{y'}{b}\right) \tag{38c}$$

where

$$\beta_m = \pi\left(m - \frac{1}{2}\right), \quad \beta_n = \pi\left(n - \frac{1}{2}\right). \tag{38d,e}$$

These expressions are readily obtained from Tables 1 and 2.

The solution of this problem uses the third and fourth terms on the RHS of equation (13)

$$T(x, y, t) = \alpha\int_{\tau=0}^{t} d\tau \int_{y'=0}^{b} \frac{q_0}{k} G_X(x, t|0, \tau) \times G_Y(y, t|y', \tau)\, dy' - \alpha\int_{\tau=0}^{t} d\tau \times \int_{x'=0}^{a} T_0 G_X(x, t|x', \tau)\left.\frac{\partial G_Y}{\partial y'}\right|_{y'=b} dx'. \tag{39}$$

Notice that the spatial integrations in equation (39) are similar in this example; the first is

$$\int_0^b G_Y\, dy' = 2\sum_{n=1}^{\infty}\frac{1}{\beta_n} e^{-\beta_n^2(\alpha(t-\tau)/b^2)}\cos\left(\beta_n\frac{y}{b}\right)(-1)^n. \tag{40a}$$

Also the integrations over τ are identical for both terms in equation (39)

$$\int_{\tau=0}^{t} e^{-\alpha(t-\tau)[\cdot]}\, d\tau = \frac{1}{\alpha[\cdot]}(1 - e^{-\alpha t[\cdot]}), \tag{40b}$$

where $[\cdot]$ is given by

$$[\cdot] = \left(\frac{\beta_m}{a}\right)^2 + \left(\frac{\beta_n}{b}\right)^2. \tag{40c}$$

Then using equations (38) and (40) in equation (39) yields the solution

$$T(x, y, t) = 4\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[1 - e^{-\alpha t[\cdot]}]\cos\left(\beta_m\frac{x}{a}\right) \times \cos\left(\beta_n\frac{y}{b}\right)(-1)^n\left\{\frac{q_0 a}{k}\frac{1}{\beta_n[\beta_m^2 + [\beta_n(a/b)]^2]} + T_0\frac{\beta_n(-1)^m}{\beta_m[(\beta_m(b/a))^2 + \beta_n^2]}\right\}. \tag{41}$$

This solution has several parts: steady state; transient; $q_0 = 0$ and $T_0 \neq 0$; and $q_0 \neq 0$ and $T_0 = 0$. The most difficult part of equation (41) to evaluate directly is the steady-state part for $T_0 \neq 0$ and $q_0 = 0$

$$T(x,y) = 4T_0 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \times \frac{\beta_n \cos\left(\beta_m \frac{x}{a}\right) \cos\left(\beta_n \frac{x}{b}\right)(-1)^{m+n}}{\beta_m\left[\left(\beta_m \frac{b}{a}\right)^2 + \beta_n^2\right]}. \quad (42)$$

See refs. [7, 11] for suggestions for using the rapidly converging transient part to aid in efficiently evaluating the steady-state part.

If the partial differential equation, equation (36), is changed to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - m^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad m^2 = \text{const.} \quad (43)$$

and the boundary conditions are the same, one can use the same GFs as given by equation (38) and the same solution as given by equation (39) except for the following changes: q_0 is replaced by $q_0 \exp(m^2\alpha\tau)$, T_0 is replaced by $T_0 \exp(m^2\alpha\tau)$, and the RHS of equation (39) is multiplied by $\exp(-m^2\alpha t)$.

7. DISCUSSION AND SUMMARY

The use of GFs has many advantages for linear transient heat conduction problems for simple geometries and homogeneous bodies. Some of the advantages are discussed in this paper. One advantage is that the GFs provide building blocks for the new numerical procedure called the surface element method [9].

A derivation of the GF equation for the transient heat conduction equation is given. An additional term which can simulate side heat loss is included. The GF equation includes the standard three linear boundary conditions plus two boundary conditions with a film of finite heat capacity.

Methods for deriving the GFs are introduced and 25 GFs are derived for boundary conditions of the first–fifth kinds. A numbering system for GFs in rectangular coordinates is given. The number system is extended in ref. [10].

The important multiplicative property is explored for rectangular and cylindrical coordinates. For rectangular coordinates the 1-D GFs can be multiplied together for all boundary conditions except the fourth and fifth kinds.

An example of the use of GFs is given.

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APPENDIX A

SEPARATION OF VARIABLES SOLUTION FOR A PLATE WITH BOUNDARY CONDITIONS OF THE FIFTH KIND

The 1-D rectangular heat conduction equation for a homogeneous plate is

$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}. \quad (A1)$$

The boundary conditions to be considered are

$$x = 0: \quad -k_1\frac{\partial T}{\partial x} + h_1 T + (\rho c b)_1 \frac{\partial T}{\partial t} = 0, \quad (A2a)$$

$$x = L: \quad k_2\frac{\partial T}{\partial x} + h_2 T + (\rho c b)_2 \frac{\partial T}{\partial t} = 0. \quad (A2b)$$

The separation of variables solution is used with

$$T = \Sigma T_m, \quad T_m = X_m(x)\theta_m(t), \quad (A3)$$

which with a separation constant of $(\beta_m/L)^2$ results in

$$X_m(x) = A_1 \sin \beta_m x/L + A_2 \cos \beta_m x/L. \quad (A4)$$

The boundary conditions, equations (A2a) and (A2b), can be given as

$$-K_1 L X'(0) + (B_1 - C_1\beta_m^2)X(0) = 0, \quad (A5a)$$

$$K_2 L X'(L) + (B_2 - C_2\beta_m^2)X(L) = 0, \quad (A5b)$$

where for $i = 1$ and 2

$$K_i = \frac{k_i}{k}, \quad B_i = \frac{h_i L}{k}, \quad C_i = \frac{(\rho c b)_i}{\rho c L}. \quad (A6a,b,c)$$

Introducing equation (A4) into equation (A5) gives two simultaneous, homogeneous equations for A_1 and A_2. These equations give the eigencondition

$$[K_1 K_2 \beta_m^2 - (B_1 - C_1\beta_m^2)(B_2 - C_2\beta_m^2)] \sin \beta_m = \beta_m[K_1(B_2 - C_2\beta_m^2) + K_2(B_1 - C_1\beta_m^2)] \cos \beta_m. \quad (A7)$$

Note that the same eigenvalues are obtained when the subscripts 1 and 2 are interchanged.

Equation (A5a) can be used to find

$$-K_1\beta_m A_1+(B_1-C_1\beta_m^2)A_2 = 0, \tag{A8}$$

which can be used to find most eigenfunctions in Table 1.

The orthogonality condition of

$$\int_0^L w(x)X_m(x)X_n(x)\,dx \begin{cases} = 0 \text{ for } m \neq n, \\ \neq 0 \text{ for } m = n, \end{cases} \tag{A9}$$

is obtained for the weighting function

$$w(x) = 1+C_1L\delta(x)+C_2L\delta(L-x), \tag{A10}$$

where $\delta(\cdot)$ is the Dirac delta function.

The norm, N_m, is given by

$$N_m = \int_0^L w(x)X_m^2(x)\,dx = A_1^2\int_0^L \sin^2\beta_m x/L\,dx$$
$$+2A_1A_2\int_0^L \sin\beta_m x/L\cos\beta_m x/L\,dx$$
$$+A_2^2\int_0^L \cos^2\beta_m x/L\,dx+C_1LA_2^2+C_2L[A_1^2\sin^2\beta_m$$
$$+2A_1A_2\sin\beta_m\cos\beta_m+A_2^2\cos^2\beta_m]. \tag{A11}$$

The solution for $T(x,t)$ for $T(x,0) = F(x)$ is then

$$T(x,t) = \int_0^L w(x)\sum_{m=0}^{\infty}\frac{1}{N_m}\,e^{-\beta_m^2(\alpha t/L^2)}X_m(x)X_m(x')F(x')\,dx'. \tag{A12}$$

APPENDIX B

PROOF OF EQUATION (30)

The LHS of equation (30) vanishes for $t \neq \tau$. At $t = \tau$, $G_2(\cdot)$ and $G_3(\cdot)$ are not continuous but each behaves like a unit step function $H(t-\tau)$ multiplied by a quantity independent of time

$$G_2(\cdot) \sim \delta(y-y')H(t-\tau), \tag{B1}$$

$$G_3(\cdot) \sim \delta(x-x')H(t-\tau). \tag{B2}$$

Since

$$\int_{-\infty}^{\infty} H^2(t-\tau)\delta(t-\tau)\,dt = \frac{H^3}{3}\bigg|_{-\infty}^{\infty} = \frac{1}{3}, \tag{B3}$$

$H^2(t-\tau)\delta(t-\tau)$ may be replaced by $\delta(t-\tau)/3$.

RESOLUTION PAR FONCTION DE GREEN DES PROBLEMES DE CONDUCTION THERMIQUE VARIABLE

Résumé—Une dérivation de solution par la fonction de Green est donnée pour l'équation linéaire de conduction thermique variable qui inclut le terme m^2T. La solution est donnée dans une forme qui inclut séparément et explicitement cinq sortes de condition aux limites. Les conditions aux limites sont les trois classiques et deux autres qui concernent un film pariétal de capacité thermique finie. Quelques suggestions sont fournies. La propriété importante multiplicative des fonctions de Green est discutée et un exemple d'utilisation des fonctions de Green est donné.

LÖSUNG INSTATIONÄRER WÄRMELEITUNGSPROBLEME MIT HILFE DER GREEN'SCHEN FUNKTION

Zusammenfassung—Es wird die Lösung der linearen instationären Wärmeleitungsgleichung, welche den Term m^2T enthält, mit Hilfe der Green'schen Funktion hergeleitet. Die Lösung ist in der Form gegeben, daß sie fünf Arten von Randbedingungen explizit und gesondert enthält. Die Randbedingungen sind die drei üblichen und zwei weitere, die den Fall eines Oberflächenfilms mit endlicher Wärmekapazität enthalten. Es werden einige Vorschläge hinsichtlich eines Numerierungssystems gemacht. Die wichtige multiplikative Eigenschaft der Green'schen Funktionen wird behandelt und ein Anwendungsbeispiel gezeigt.

РЕШЕНИЕ ЗАДАЧ НЕСТАЦИОНАРНОЙ ТЕПЛОПРОВОДНОСТИ С ПОМОЩЬЮ ФУНКЦИЙ ГРИНА

Аннотация—Предложено решение с помощью функций Грина уравнения линейной нестационарной теплопроводности, включающего слагаемое m^2T. При решении в явном виде используются пять граничных условий, из которых три являются стандартными и два дополнительными для поверхностного слоя конечной теплоемкости. Приведены некоторые соображения по поводу размерности системы. Проведено обсуждение важного свойства мультипликативности функций Грина и на примере показано их использование.